THE FIRST REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE first regular meeting of the Southwestern Section of the AMERICAN MATHEMATICAL SOCIETY took place in St. Louis, Mo., on the 30th of November, 1907. About fifty persons, including the following members of the Society, were present:

Professor L. D. Ames, Mr. Charles Ammerman, Professor Florian Cajori, Professor A. S. Chessin, Professor C. E. Comstock, Brother Constantius, Professor S. C. Davisson, Professor G. R. Dean, Professor G. W. Droke, Dr. Otto Dunkel, Professor B. F. Finkel; Professor G. B. Halsted, Mr. W. W. Hart, Professor H. C. Harvey, Professor E. R. Hedrick, Dr. Louis Ingold, Dr. G. O. James, Professor O. D. Kellogg, Professor G. A. Miller, Professor H. B. Newson, Professor J. H. Scarborough, Professor J. B. Shaw, Mr. R. L. Short, Professor H. E. Slaught, Dr. Clara E. Smith, Mr. H. P. Stellwagen, Mr. E. H. Taylor, Professor C. A. Waldo, Professor B. M. Walker, Dr. Paul Wernicke.

The meeting was called to order by Professor Hedrick at 10:30 A. M., at the McKinley High School. The morning session adjourned at 12:30 P. M. The afternoon session was called to order at 3 P. M., at Washington University. The reading of papers was preceded by a short business meeting. Professor Chessin (chairman), Professor Kellogg (secretary) and Professor Newson were elected to serve on the program committee for the ensuing year. The University of Kansas was unanimously selected as the next meeting place of the Southwestern Section.

The following papers were read:

(1) Dr. P. WERNICKE: "On Euler's tactical '36 officers' problem."

(2) Dr. P. WERNICKE: "Extension of the map-color theorem to 3 dimensional space."

(3) Professor W. F. OSGOOD: "On the differentiation of definite integrals."

(4) Professor F. CAJORI: "Notes on the history of the slide rule."

(5) Dr. LOUIS INGOLD: "Note on areal cross ratios."

(6) Dr. G. O. JAMES: "A relation connecting aberration and parallax" (preliminary report).

(7) Professor J. B. SHAW: "A new graphical method for quaternions."

(8) Professor E. R. HEDRICK: "On a definition of the jacobian" (preliminary report).

(9) Professor A. S. CHESSIN: "On an integral appearing in photometry."

(10) Professor O. D. KELLOGG: "Real roots of an algebraic equation."

(11) Dr. LOUIS INGOLD: "Note on a connection between algebraic invariants and the invariants of a differential form."

(12) Professor H. B. NEWSON: "On the resultant of two collineations."

(13) Professor G. A. MILLER: "On the holomorph of the cyclic group of order p^m."

(14) Professor E. W. DAVIS: "Colored imaginaries on a cubic."

In the absence of the authors, Professor Osgood's paper was read by Professor Hedrick and Professor Davis's paper was read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Euler and Cayley have given some attention to the problem of arranging in a square of n rows and columns n^2 officers of n different regiments, the n of each regiment being of n different ranks. Every rank and regiment is to be represented in each row or column. Such arrays are possible unless $n \equiv 2 \pmod 4$. No proof is extant of their impossibility in this particular case. Dr. Wernicke attempted such a proof on the principle that the four classifications involved (viz. into rows, columns, officers' rank and regiments) are tactically equivalent. Hence applications become possible to more complex problems involving an interference of classifications.

2. Dr. Wernicke defines a "three-dimensional map" to be a division of three-dimensional space by any number of surfaces meeting in lines only. The latter meet in points only, which we call the vertices of the map. The portions of the lines extending from any vertex to the consecutive one are the edges; the portions of the surfaces bounded by the edges, the partitions; the portions of space partitioned off, the cells or districts of the map. If a map contains neither multiply-connected cells nor partitions, its cells may be filled with nine different gases in such a way

that no partition separates cells holding the same gas. The method of proof is similar to that of the corresponding four-color theorem of plane maps.

3. In this paper Professor Osgood gives an account of the existing proofs of the formulas for differentiating a definite integral with respect to a parameter, followed by a critique of these proofs. He then gives a new proof based on Green's theorem, which is free from some of the objections which may be brought against the older proofs. The paper will be offered to the *Annals of Mathematics*.

4. Charles Hutton attributed the invention of the straight-edge slide rule to Edmund Wingate, but gave no references to Wingate's publications. Augustus De Morgan attributed the invention to William Oughtred and denied that Wingate ever wrote on the slide rule. Professor Cajori was able to quote from one of Wingate's works of 1628 and to show that Wingate was the first inventor. The "runner," often attributed to Mannheim (1850), was shown to be the invention of I. Newton and E. Stone. Probably the first inversion of a logarithmic line is found in Everard's slide rule, used in England in gauging during the latter part of the seventeenth and in the eighteenth century. Newton's and Stone's adaption of the slide rule to the solution of numerical equations was explained. Finally, the time of the introduction of the slide rule into the United States was considered.

5. Dr. Ingold considers the ratio of areas of triangles formed from six points, A, B, C, P, Q, R, in a plane. The double ratio $[PA, BC, QR]$ is defined to be

$$\frac{PBC}{ABC} \div \frac{PQR}{AQR},$$

which can be proved to be invariant under projection. The possible double ratios are seen to be equal in sets of eight; there remain therefore ninety which are distinct. From identities such as $ABC.PQR = PBC.AQR + QBC.PAR + RBC.PQA$ a number of relations are easily established among the different double ratios, so that not more than four of the ninety are independent, the others being expressible in terms of these.

6. In any system of spherical coordinates the aberrational corrections may be computed from the corresponding formulas for the parallaxes by differentiating these formulas with respect to the time, treating the angular coordinates as constant. Dr. James obtains the following results:

$$\Delta A_a = \frac{d}{dt}\left(\frac{rP}{\cos B - Q \sin B + R \cos B}\right)$$

$$\Delta B_a = \frac{drQ}{dt} - \frac{drP}{dt} \sin B \tan \frac{\Delta A_a}{2}.$$

7. Any quaternion $q = w + xi + yj + zk$ may be written in the form $q = w + xi + (y + zi)j$. We may represent a quaternion graphically therefore by representing $w + xi$ and $y + zi$ by the ordinary method of representing complex numbers, $w + xi$ being drawn on one plane and $y + zi$ on a second plane, called the j-plane. For convenience we may divide an ordinary plane into four quadrants, the first quadrant being the plane of $w + xi$, the third the quadrant of $y + zi$. The second and fourth quadrants, by orthogonal projections, furnish also representations for $x + yk$, and $w + zk$. The product and the sum of two quaternions may be found graphically by methods similar to those in use in the theory of complex numbers. The theory of two independent complex variables finds a graphical representation here also. Finally the descriptive geometry of four dimensional space and the quaternion algebra as applied to four-dimensional space come into intimate contact, and analytic processes may be easily interpreted in geometric procedures, or conversely. The paper will be offered to the *Transactions*.

8. In this paper, Professor Hedrick offers a modification of the definition of the jacobian suggested by Jordan (Cours d'analyse volume 1, § 148) and by Porter (*L'Enseignement mathématique*, volume 9, part 4, page 272). The jacobian is defined as the limit of the quotient of two areas, one of which is a right triangle in the original plane. This limit exists whenever the ordinary jacobian exists, and á fortiori whenever the limit used by Jordan and Porter exists. Certain advantages are pointed out.

9. Professor Chessin's paper appears in full elsewhere in the present number of the BULLETIN.

10. By a theorem of Professor Van Vleck (*Annals of Mathematics*, July, 1903), an algebraic equation with real coefficients $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n = 0$ will have all or all but one of its roots (according as n is even or odd) imaginary if all the members of a certain set of determinants are positive. Professor Kellogg deduces from Descartes's rule of signs a necessary condition that all the roots of such an equation be real; namely that all the determinants $c_{j-1}c_{j+1} - c_j^2 < 0$ ($k < j < n$, where c_k is the first coefficient different from 0). A second, sharper condition, of which the first is a consequence, is that none of the quadratics

$$\begin{vmatrix} x^2 & x & 1 \\ c_{j-1} & c_j & c_{j+1} \\ c_{j+1} & c_{j+1} & c_{j+2} \end{vmatrix} = 0 \qquad (k \leqq j < n,\ c_{-1} = c_{n+1} = 0)$$

has real roots. The paper will appear in the January number of the *Annals of Mathematics*.

11. In Dr. Ingold's second paper a quadratic differential form D with constant coefficients is considered. The differential parameters of such a form which involve no functons except algebraic forms $b, c, \cdots$ are concomitants of these forms and of the quadratic algebraic form a having the same coefficients as D. Conversely, any concomitant of the system $a, b, c, \cdots$, when multiplied by a power of the discriminant of D, becomes a differential parameter of D. Relations among the differential parameters of D, when thus specialized, yield relations among the concomitants of the system $a, b, c, \cdots$.

12. Professor Newson's paper seeks to determine invariants of collineations. As one such is found a certain cross ratio, and it is shown that this cross ratio for the product of two collineations is a simple function of the two corresponding cross ratios for the two collineations determining the product.

13. The paper by Professor Miller is a continuation of his article entitled "On the holomorph of a cyclic group" which was published in the *Transactions*, volume 4 (1903), page 151. While the holomorph of a cyclic group plays a fundamental role in many group theory considerations, it also involves all questions in regard to the exponent to which numbers belong with respect to a given arbitrary modulus m. For instance,

the numbers which Epstein, in his recent article published in the *Archiv der Mathematik und Physik*, calls primitive roots of m are those which correspond to the operators of highest order in the group of isomorphisms of the cyclic group of order m, and hence the determination of the number of such primitive roots is a very special case of the determination of the number of operators of a given order in an abelian group. In the present paper special attention is paid to the group of isomorphisms of the holomorph of the cyclic group of order 2^m, and one of the most important results is stated as follows: The group of isomorphisms of the holomorph of the cyclic group of order 2^m is the direct product of the group of order 2 and the group of cogredient isomorphisms of the double holomorph of the cyclic group of order 2^m. This paper will appear in the *Transactions*.

14. In this paper Professor Davis studies the connection between his theory of colored imaginaries and the pole and polar theory of the cubic curve. If $f(x, y, z) = 0$ is the equation of the cubic, $\Delta'(x, y, z)$ the first polar of x', y', z' with regard to $f(x, y, z)$, while Δ'' is the polar of x'', y'', z'', then when we write $x = x' + ix''$, etc., we get $f(x', y', z') = \Delta'(x'', y'', z'')$ and $f(x'', y'', z'') = \Delta''(x', y', z')$. It is upon the basis of these two equations that the paper is built.

O. D. KELLOGG,
Secretary of the Section.

NOTE ON THE COMPOSITION OF FINITE ROTATIONS ABOUT PARALLEL AXES.

BY PROFESSOR ALEXANDER ZIWET.

1. IT is well known that the succession of two finite rotations of a rigid plane figure in its plane (or, what amounts to the same, of a rigid body about parallel axes), say a rotation of angle θ' about a point O' followed by a rotation θ'' about O'', is equivalent to a single rotation of angle $\theta = \theta' + \theta''$ about a point O. The center O is found as the intersection of the lines obtained by turning $O'O''$ about O' through an angle $-\frac{1}{2}\theta'$ and $O''O'$ about O'' through $+\frac{1}{2}\theta''$.

As a clockwise rotation of angle ϕ is equivalent to a counter-clockwise rotation of angle $2\pi - \phi$, the angles of rotation can

all be made of the same sense, say counterclockwise; and as a rotation of angle $\phi+2\kappa\pi$ is equivalent to the rotation of angle ϕ, the angles can be confined to values between 0 and 2π.

The construction of the point O is therefore always possible unless $\theta=\theta'+\theta''$ is $\equiv 0 \pmod{2\pi}$. In this limiting case the center lies at infinity, and the resultant displacement is a translation whose vector is readily determined.

2. When there are more than two successive rotations, the resultant of the first and second rotations can be compounded by the same method with the third rotation, and so on. It is apparent that the angle of the final resultant rotation is the sum of the angles of the given rotations, but the construction of the center becomes rather complicated. It is the object of the present note to indicate a convenient method for finding this center, by applying vector methods systematically.

3. A single rotation, θ' about O' (Fig. 1), carries any point A of the rigid figure from the initial position A_0 to the final

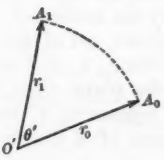

FIG. 1.

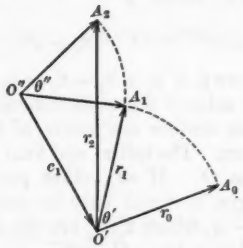

FIG. 2.

position A_1. Taking the fixed point O' as origin and putting $A_0-O'=\mathbf{r}_0$, $A_1-O'=\mathbf{r}_1$, we have the vector equation

$$\mathbf{r}_1=e^{i\theta'}\mathbf{r}_0,$$

which means that the operator $e^{i\theta'}$ applied to the vector $\mathbf{r}_0$ turns it in the plane through the angle θ'.*) This is of course merely another form of stating the ordinary method of complex numbers.

Two successive rotations, θ' about O' and θ'' about O'' (Fig. 2), carry the point A from A_0 to A_1 and from A_1 to A_2. With $O'-O''=\mathbf{c}_1$, $A_2-O'=\mathbf{r}_2$ we have $A_1-O''=\mathbf{r}_1+\mathbf{c}_1$,

*G. Peano, Gli elementi di calcolo geometrico, Torino, 1891, p. 19.

$A_2 - O'' = \mathbf{r}_2 + \mathbf{c}_1$, whence

$$\mathbf{r}_2 + \mathbf{c}_1 = e^{i\theta''}(\mathbf{r}_1 + \mathbf{c}_1),$$

or, replacing $\mathbf{r}_1$ by its value from the preceding equation,

$$\mathbf{r}_2 = e^{i(\theta''+\theta')}\mathbf{r}_0 + e^{i\theta''}\mathbf{c}_1 - \mathbf{c}_1.$$

In the case of three successive rotations, θ' about O', θ'' about O'', θ''' about O''', putting $O'' - O''' = \mathbf{c}_2$, $A_3 - O' = \mathbf{r}_3$ we have

$$\mathbf{r}_3 + \mathbf{c}_1 + \mathbf{c}_2 = e^{i\theta'''}(\mathbf{r}_2 + \mathbf{c}_1 + \mathbf{c}_2),$$

whence eliminating $\mathbf{r}_2$ we find

$$\mathbf{r}_3 = e^{i(\theta'''+\theta''+\theta')}\mathbf{r}_0 + e^{i(\theta'''+\theta'')}\mathbf{c}_1 + e^{i\theta'''}\mathbf{c}_2 + \mathbf{c}_3,$$

where $\mathbf{c}_3 = -\mathbf{c}_1 - \mathbf{c}_2 = O''' - O'$.

Similarly we find for four successive rotations, θ' about O', $\cdots$, θ^{iv} about O^{iv}

$$\mathbf{r}_4 = e^{i(\theta^{iv}+\theta'''+\theta''+\theta')}\mathbf{r}_0 + e^{i(\theta^{iv}+\theta'''+\theta'')}\mathbf{c}_1 + e^{i(\theta^{iv}+\theta''')}\mathbf{c}_2 + \mathbf{c}^{i\theta^{iv}}\mathbf{c}_3, + \mathbf{c}_4,$$

where $\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_4 = 0$. The vectors $\mathbf{c}_1, \cdots, \mathbf{c}_4$ are equal to the sides of the quadrilateral formed by the centers $O', \cdots, O^{iv}$; these centers are points of the fixed plane, not of the moving figure. The initial and final radii vectors $\mathbf{r}_0$, $\mathbf{r}_4$ of A are drawn from O'. If any other point Q of the plane were taken as origin, it would only be necessary to replace $\mathbf{r}_0$, $\mathbf{r}_4$ by $\mathbf{r}_0' - \mathbf{q}$, $\mathbf{r}_4' - \mathbf{q}$, where $\mathbf{r}_0'$, $\mathbf{r}_4'$ are the radii vectors of A drawn from Q, and $\mathbf{q} = O' - Q$.

4. In the expression found for $\mathbf{r}_4$, all terms after the first are independent of the particular point A of the figure. Their sum represents a vector $\mathbf{s}_4$ which can be written

$$\mathbf{s}_4 = e^{i\theta^{iv}}(e^{i\theta'''}(e^{i\theta''}\mathbf{c}_1 + \mathbf{c}_2) + \mathbf{c}_3) + \mathbf{c}_4.$$

This form indicates the most convenient way of constructing the vector $\mathbf{s}_4$ (Fig. 3): turn $O' - O'' = \mathbf{c}_1$ about O'' through θ'' and add $\mathbf{c}_2$; then turn the sum so obtained about O''' through θ''' and add $\mathbf{c}_3$; finally turn the vector so obtained about O^{iv} through θ^{iv} and add $\mathbf{c}_4$; this gives the vector $\mathbf{s}_4 = O_4 - O'$ which evidently represents the displacement of that point of the rigid figure which originally coincided with O'.

Denoting the angle of the resultant rotation by θ, so that $\theta \equiv \theta' + \theta'' + \theta''' + \theta^{iv} (\mathrm{mod}\ 2\pi)$ and introducing the vector $\mathbf{s}_4$, we have the simple result

$$\mathbf{r}_4 = e^{i\theta}\mathbf{r}_0 + \mathbf{s}_4.$$

5. It is obvious that in the case of n successive rotations we have similarly

$$\mathbf{r}_n = e^{i\theta}\mathbf{r}_0 + \mathbf{s}_n,$$

where $\theta \equiv \theta' + \theta'' + \cdots + \theta^{(n)}$ and

$$\mathbf{s}_n = e^{i\theta^{(n)}}\{\cdots [e^{i\theta'''}(e^{i\theta''}\mathbf{c}_1 + \mathbf{c}_2) + \mathbf{c}_3] + \cdots + \mathbf{c}_{n-1}\} + \mathbf{c}_n.$$

Thus, the final radius vector of any point of the rigid figure is found by turning its initial radius vector about O' through an

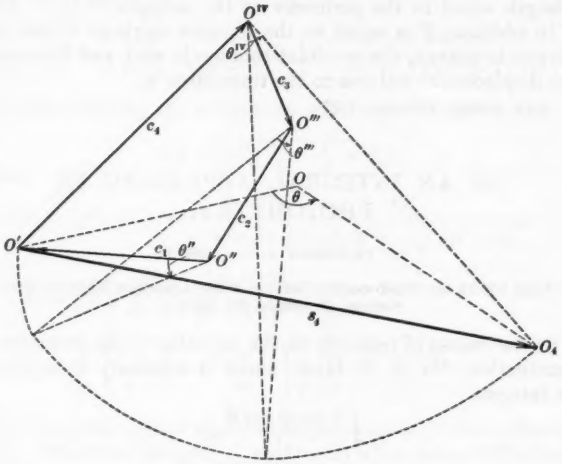

FIG. 3.

angle $\theta \equiv \theta' + \theta'' + \cdots + \theta^{(n)}$ and adding to it the vector $\mathbf{s}_n$. In other words, the resultant displacement is resolved into the rotation of angle θ about O' and the translation $\mathbf{s}_n$ of O'.

The center O of the equivalent single rotation being a fixed point its radius vector $\mathbf{r}$ is found by putting $\mathbf{r}_n = \mathbf{r}_0 = \mathbf{r}$ which

giv

$$(1 - e^{i\theta})\mathbf{r} = \mathbf{s}_n, \quad \text{or} \quad \mathbf{r} = \frac{1}{2 \sin \frac{1}{2}\theta} e^{i(\pi-\theta)/2}\mathbf{s}_n.$$

This means that the point O is the vertex of the isosceles triangle whose base is $O'O_n$ and whose angle at O is θ if $\theta < \pi$ and is $2\pi - \theta$ if $\theta > \pi$. In the former case the sense $O'O_nO$ is positive, in the latter it is negative. If $\theta = \pi$, the resultant displacement is a reversal (Umwendung) about the midpoint of $O'O_n$. If $\theta \equiv 0 \pmod{2\pi}$ the resultant displacement reduces to the translation $\mathbf{s}_n$.

If, in particular, the angles of rotation θ'', θ''', $\cdots$, $\theta^{(n)}$ are equal, respectively, to the exterior angles of the polygon of centers at O'', O''', $\cdots$, $O^{(n)}$, the vector $\mathbf{s}_n$ has, as appears at once from its construction (Fig. 3), the direction and sense of $\mathbf{c}_n$ and a length equal to the perimeter of the polygon $O'O''\cdots O^{(n)}$. If, in addition, θ' is equal to the exterior angle at O' and the polygon is convex, the resultant rotation is zero, and the resultant displacement reduces to the translation $\mathbf{s}_n$.

ANN ARBOR, *December*, 1907.

ON AN INTEGRAL APPEARING IN PHOTOMETRY.

BY PROFESSOR A. S. CHESSIN.

(Read before the Southwestern Section of the American Mathematical Society, November 30, 1907.)

IN the course of research on the variation of the intensity of illumination, Mr. E. P. Hyde found it necessary to evaluate the integral

$$\int \frac{i \cos\phi \cos\theta}{r^2} dS$$

taken over a part of the surface of a cylinder.† Having encountered some difficulties on the mathematical side of the problem, Mr. Hyde requested the author to compute the value of this integral. As this computation contains some features of interest to mathematicians the solution is here presented

† "Talbot's law as applied to the rotating sectored disc," by E. P. Hyde, *Bulletin of the U. S. Bureau of Standards*, vol. 2, No. 1.

without, however, referring to the physical aspects of the problem. The purely mathematical problem, then, may be stated as follows:

Evaluate the integral

$$J = ia \int_{-h}^{h} dy \int_{-\cos^{-1} a/l}^{\cos^{-1} a/l} \frac{(a - l\cos\alpha)(l - a\cos\alpha)d\alpha}{(a^2 + l^2 - 2al\cos\alpha + y^2)^2}.$$

By means of the substitutions

$$\tan \tfrac{1}{2}\alpha = x, \quad \frac{(l+a)^2 + y^2}{(l-a)^2 + y^2} = z,$$

$$\frac{l+a}{l-a} = p, \quad \frac{(l+a)^2 + h^2}{(l-a)^2 + h^2} = q,$$

the given integral is transformed into

$$i\left(1 - \frac{a}{l}\right) \int_{p^2}^{q} \sqrt{\frac{z-1}{p^2 - z}}\, dz \int_{0}^{1/\sqrt{p}} \frac{(p^2x^4 - 1)dx}{(x^2+1)(zx^2+1)^2}$$

and, after effecting the integration with regard to x, becomes

$$i(1 - a/l)(J_1 + J_2 + J_3),$$

where

$$J_1 = (p^2 - 1)\tan^{-1}\frac{1}{\sqrt{p}} \int_{p^2}^{q} \frac{dz}{(z-1)\sqrt{(z-1)(p^2 - z)}},$$

$$J_2 = \tfrac{1}{2}\sqrt{p} \int_{p^2}^{q} \frac{(p-z)dz}{z\sqrt{(z-1)(p^2 - z)}},$$

$$J_3 = \int_{p^2}^{q} \left[\frac{z - p^2}{z-1} - \frac{p^2 + z^2}{2z}\right] \frac{\tan^{-1}\sqrt{z/p}\, dz}{\sqrt{z(z-1)(p^2 - z)}}.$$

The integrals J_1 and J_2 are readily evaluated by the ordinary rules. The third integral J_3 apparently offers some difficulties. The integration, however, is very simple and does not even require the introduction of elliptic functions.

In fact, consider the integral

$$U = \int \left[\frac{z - p^2}{z-1} - \frac{p^2 + z^2}{2z}\right] \frac{dz}{\sqrt{z(z-1)(p^2 - z)}}.$$

It belongs to a class of pseudo-elliptic integrals discussed by

Euler,* Raffy,† and Goursat.‡ Adopting Goursat's method, we have in this case three relations of involution by which the roots of the equation $z(z-1)(p^2-z)=0$ are permuted in pairs. Two of these relations are

$$z'z''=p^2, \quad z'z''-(z'+z'')+p^2=0.$$

On the other hand, if we put

$$\frac{z-p^2}{z-1}-\frac{p^2+z^2}{2z}=f_1(z)+f_2(z),$$

where

$$f_1(z)=\frac{z-p^2}{z-1}-z, \quad f_2(z)=\tfrac{1}{2}\left(z-\frac{p^2}{z}\right),$$

we note that each one of the functions f_1 and f_2 satisfies a relation, corresponding to one or the other of the two relations of involution given above, viz.,

$$f_1(z)+f_1\left(\frac{p^2-z}{1-z}\right)\equiv 0, \quad f_2(z)+f_2\left(\frac{p^2}{z}\right)\equiv 0.$$

Hence,§ the integral U is pseudo-elliptic.

At the same time, if we apply the author's reduction formulas given in the pages of this BULLETIN,‖ viz., formulas (5) and (6), where we should put $m=r=3$, $\Sigma n_k=s=\rho=2$, and, consequently, $q=1$, we find that

$$U=\frac{(Az+B)\sqrt{z(z-1)(p^2-z)}}{z(z-1)}+\lambda_0\int\frac{dz}{\sqrt{z(z-1)(p^2-z)}}+\lambda_1\int\frac{zdz}{\sqrt{z(z-1)(p^2-z)}}.$$

But the two integrals on the right-hand side are obviously elliptic. Hence, since it has been shown that U is pseudo-

* "Sur une formule d'Euler," par M. Hermite, *Journal de Liouville*, 1880.

† "Sur les transformations invariantes des différentielles elliptiques," par L. Raffy, *Bulletin de la Soc. Math. de France*, 1884.

‡ "Sur les integrales pseudo-elliptiques," par E. Goursat, *Bulletin de la Soc. Math. de France*, 1885. See also Goursat, Cours d'analyse, vol. 1, pp. 264–266.

§ L. c.

‖ "Note on hyperelliptic integrals," BULLETIN, vol. 4, pp. 93–96.

elliptic, we must have $\lambda_0 = \lambda_1 = 0$, so that, barring a constant of integration,

$$U = (Az + B)\sqrt{\frac{p^2 - z}{z(z-1)}}.$$

It now only remains to determine the coefficients A and B. This is readily done by comparing both sides of the identity

$$\frac{z-p^2}{z-1} - \frac{p^2+z^2}{2z} = \sqrt{z(z-1)(p^2-z)}\,\frac{d}{dz}\left[(Az+B)\sqrt{\frac{p^2-z}{z(z-1)}}\right]$$

which gives $A = B = 1$ and, therefore, barring the constant of integration,

$$U = (z+1)\sqrt{\frac{p^2-z}{z(z-1)}}.$$

Returning now to the integration of J_3 we observe that

$$J_3 = \int_{p^2}^{q} \tan^{-1}\sqrt{\frac{z}{p}}\,dU = \left[U\tan^{-1}\sqrt{\frac{z}{p}}\right]_{p^2}^{q} - \tfrac{1}{2}\sqrt{p}\int_{p^2}^{q}\frac{U dz}{(z+p)\sqrt{z}}$$

$$= (q+1)\sqrt{\frac{p^2-q}{q(q-1)}}\tan^{-1}\sqrt{\frac{q}{p}} + \tfrac{1}{2}\sqrt{p}\int_{p^2}^{q}\frac{(z+1)(z-p^2)dz}{z(z+p)\sqrt{(z-1)(p^2-z)}}$$

or, adding the integral J_2,

$$J_2 + J_3 = (q+1)\sqrt{\frac{p^2-q}{q(q-1)}}\tan^{-1}\sqrt{\frac{q}{p}} - \tfrac{1}{2}(p^2-1)\sqrt{p}\int_{p^2}^{q}\frac{dz}{(z+p)\sqrt{(z-1)(p^2-z)}}$$

It is a simple matter now to complete the integration. The result is *

$$J = \frac{i}{l}\left\{2a\tan^{-1}\frac{h}{\sqrt{l^2-a^2}} + h\left(\sqrt{q} + \frac{1}{\sqrt{q}}\right)\tan^{-1}\sqrt{\frac{q}{p}} - 2h\cot^{-1}\sqrt{p}\right\}.$$

SAINT LOUIS,
November 30, 1907.

* It is given in Mr. Hyde's paper in a slightly different but equivalent form. L. c., p. 8.

HERMITIAN FORMS WITH ZERO DETERMINANT.

BY PROFESSOR J. I. HUTCHINSON.

(Read before the American Mathematical Society, December 28, 1907.)

VERY little has yet been attempted in the arithmetic determination of infinite discontinuous groups on two or more variables. The earliest memoir on this subject, by Picard,* treats of groups which leave certain ternary Hermitian forms invariant. Such forms can always be reduced, as Picard has remarked in a later paper,† to the form

$$\alpha u\bar{u} + \beta v\bar{v} - \gamma w\bar{w}, \qquad (\alpha, \beta, \gamma > 0), \tag{1}$$

if the determinant of the form is negative. The locus of singular (or limit) points for the group is the spread obtained by equating (1) to zero. It is the object of the present note to determine the kind of groups which have their limit points on such a quadric spread in case the corresponding Hermitian form has a zero determinant.

Let the given form f be represented by

$$f \equiv \sum_{i,\,k=1}^{n} f_{ik} x_i \bar{x}_k, \tag{2}$$

in which $\bar{x}$ is the conjugate of x, and the coefficient f_{ik} is the conjugate of f_{ki}. The determinant $D = |f_{ik}|$ of this form we assume to be zero and of rank k. Accordingly the elements of a certain column in D (the gth, say) are a linear combination of the corresponding elements of the other columns, that is,

$$f_{ig} = \sum_{\lambda} \bar{a}_\lambda f_{i\lambda} \qquad \left(\begin{smallmatrix}\lambda=1,2,\cdots,g-1,g+1,\cdots,n\\ i=1,2,\cdots,n\end{smallmatrix}\right),$$

in which the $\bar{a}_\lambda$ are finite and g is a suitably chosen integer.

If now we make the substitution

$$\begin{aligned} \eta_\lambda &= x_\lambda + a_\lambda x_g \quad (\lambda = 1, 2, \cdots, g-1, g+1, \cdots, n), \\ \eta_g &= x_g, \end{aligned} \tag{3}$$

* "Sur une classe de groupes discontinues de substitutions linéaires, etc.," *Acta Mathematica*, vol. 1 (1881), p. 297.

† "Sur les formes quadratiques ternaires indéfinies à indéterminées conjuguées, etc.," *Acta Mathematica*, vol. 5 (1885), p. 121.

the form f reduces to

$$(4) \qquad \sum_{\lambda,\mu} f_{\lambda\mu}\eta_\lambda\bar{\eta}_\mu \quad (\lambda, \mu = 1, 2, \cdots, g-1, g+1, \cdots, n),$$

as may readily be verified by substituting in (4) the preceding expressions for η_λ and using the relations

$$\sum_\lambda f_{\lambda\mu}a_\lambda = \sum_\lambda \bar{f}_{\mu\lambda}a_\lambda = \bar{f}_{\mu g},$$

$$\sum_{\lambda\mu} f_{\lambda\mu}a_\lambda\bar{a}_\mu = \sum_\lambda a_\lambda f_{\lambda g} = \bar{f}_{gg} = f_{gg}.$$

The determinant of (4) is the first minor F_{gg} corresponding to f_{gg} in D. If the rank k of D is less than $n-1$, F_{gg} is zero and the process is continued until we reach a form whose determinant $M_{\mu\mu}$ does not vanish. This determinant is evidently a diagonal minor of D of order k, and accordingly its elements are common to the rows and columns nnmbered $\mu_1, \mu_2, \cdots, \mu_k$, these k integers being selected from the set $1, 2, \cdots, n$.

It is clear that by choosing different values for g in the transformation (3), and in the subsequent ones, the reduction could be made in a variety of ways and the determinant of the final form may be any non-vanishing diagonal minor of D of order k. We have now to show that, *in whatever way the given Hermitian form is reduced, the determinant of the final form always has the same sign.* This follows from a well-known theorem in determinants. For, let $M_{\nu\nu}$ be the diagonal minor of order k whose elements are taken from the rows and columns numbered $\nu_1, \nu_2, \cdots, \nu_k$. Denote by $M_{\mu\nu}$ the minor

$$\begin{vmatrix} a_{\mu_1\nu_1} & a_{\mu_1\nu_2} & \cdots & a_{\mu_1\nu_k} \\ a_{\mu_2\nu_1} & a_{\mu_2\nu_2} & \cdots & a_{\mu_2\nu_k} \\ \cdot & \cdot & \cdot & \cdot \\ a_{\mu_k\nu_1} & \cdots & & a_{\mu_k\nu_k} \end{vmatrix}$$

and by $M_{\nu\mu}$ the minor obtained from this by interchanging μ_i and ν_i $(i = 1, 2, \cdots, k)$. By interchanging rows and colums in $M_{\nu\mu}$, it is readily seen to be the numerical equivalent of $\bar{M}_{\mu\nu}$.

But since the determinant of the four minors thus obtained is zero,* we have

$$M_{\mu\mu}M_{\nu\nu} = M_{\mu\nu}\overline{M}_{\mu\nu}. \tag{5}$$

The right member is positive or zero and hence if $M_{\mu\mu}$ and $M_{\nu\nu}$ be not zero they must have the same sign.

Suppose now that (2) has been reduced to the form

$$\sum_{i,j=n-k+1}^{n} f_{ij}y_i\bar{y}_j, \tag{6}$$

the determinant of which $|f_{ij}|$ is not zero. A transformation on the x's which leaves the form (1) unaltered will be replaced by a corresponding transformation on the y's of the form

$$\begin{aligned} y'_\lambda &= \sum_{\mu=\lambda}^{n} a_{\lambda\mu}y_\mu \qquad (\lambda = 1, 2, \cdots, n-k), \\ y'_i &= \sum_{j=n-k+1}^{n} a_{ij}y_j \quad (i = n-k+1, \cdots, n). \end{aligned} \tag{7}$$

A closer characterization of the group can be made sufficiently clear by an examination of the simplest case, $n = 3$. The general substitution of the group may be represented by

$$x' = ax + by + cz, \quad y' = ay + \beta z, \quad z' = \gamma y + \delta z. \tag{8}$$

Consider first the substitutions of the form

$$x' = a'x + b'y + c'z, \quad y' = y, \; z' = z, \tag{9}$$

the totality of which form a subgroup of (8). As a group of transformations on the single variable x, it must be contained in the cyclic or the parabolic rotation groups.† It follows then that either b', c' are zero, or that a can have only the values ± 1 and the remaining terms in x' are expressible in the form

$$b'y + c'z = m\omega + n\omega'$$

* E. Pascal, Die Determinanten, Leipzig, 1900, p. 195. The determinant D might be called skew conjugate, and includes the symmetric and the skew-symmetric (after multiplying all the elements by i) as special cases. The theorem formulated in (5) would specialize in these cases as follows:

If a symmetric, or a skew-symmetric, determinant of order n is of rank $k < n$, then all diagonal minors of order k which do not vanish have the same sign.

† Fricke-Klein, Automorphe Functionen, I, p. 214 ff.

in which ω, ω' are arbitrary, fixed parameters and m, n are integers. Hence, *on replacing* y, z *by suitable variables* ω, ω', *the group* (8) *reduces to one of the forms*

$$(10) \qquad \begin{vmatrix} a & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \delta & \gamma \end{vmatrix}, \qquad \begin{vmatrix} \pm 1 & m & n \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{vmatrix}.$$

If the group is of the first type, it is completely reducible * and of no interest as a ternary group. If of the second type, the coefficients $\alpha, \beta, \gamma, \delta$ must be integers such that $\alpha\delta - \beta\gamma = 1$, otherwise the combination of two substitutions would give one whose determinant does not have integers in its first row, and the group would contain infinitesimal transformations.†

The method of reasoning which we have applied to ternary groups is readily extended to the case of n variables and gives the result:

Either the group is completely reducible and has the form ‡

$$\begin{vmatrix} G & 0 \\ 0 & G' \end{vmatrix}$$

in which G' *is the group of the reduced form* (6) *and* G *is any discontinuous group in the remaining* $n - k$ *variables, or it has the form*

$$\begin{vmatrix} G_{11} & 0 & 0 \\ 0 & G_{22} & G_{23} \\ 0 & 0 & G_{33} \end{vmatrix},$$

where G_{11} *is any discontinuous group on* p *of the variables* $(p < n - 2)$, *and the other matrices have the form*

*See W. Burnside, *Acta Mathematica*, vol. 28 (1904), p. 369, and A. Loewy, *Transactions*, vol. 6 (1905), p. 505.

†The group thus determined is of particular interest on account of its connection with the theory of elliptic functions. In fact, the Weierstrass functions constitute a complete system of invariants for the group. See Klein-Fricke, Modulfunctionen, vol. 2, p. 3 ff.

‡For this very convenient symbolic representation see A. Loewy, *Transactions*, vol. 4 (1903), p. 44.

$$G_{22} = \begin{vmatrix} \pm 1 & 0 & 0 & \cdots \\ 0 & \pm 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \pm 1 \end{vmatrix}, \quad G_{23} = \begin{vmatrix} 0 & 0 & \cdots & m_1 & n_1 \\ 0 & 0 & \cdots & m_2 & n_2 \\ & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & m_r & n_r \end{vmatrix},$$

$$G_{33} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix},$$

in which m_i, n_i $\alpha, \beta, \gamma, \delta$ *are integers and* $\alpha\delta - \beta\gamma = 1$.

CORNELL UNIVERSITY,
December, 1907.

TWO TETRAEDRON THEOREMS.

BY PROFESSOR HENRY S. WHITE.

THE sphere and the tetraedron yield two combinations familiar to students of geometry, those in which one object is inscribed in the other; and one less well known, that in which the edges of the tetraedron are tangents to the sphere. A novel theorem upon the circumscribed tetraedron was propounded in 1897 by Bang and proved by Gehrke, and has been made the starting-point for extended developments by Franz Meyer * (1903) and Neuberg (1907).† It is this: If the contact point in each face of a tetraedron circumscribed about a sphere be joined by a straight line to each vertex in its face, then three angles at each contact point are equal respectively to the three formed at any other contact point. Or it may be stated thus: Opposite edges of a circumscribed tetraedron subtend equal angles at the points of contact of the faces which contain them.

While elementary proofs of this are interesting, a more elaborate deduction is of value here as suggesting a second theorem. It can be made to depend upon the well-known theorem from the projective geometry of a straight line, namely,

* W. Franz Meyer: "Ueber Verallgemeinerungen von Sätzen über die Kugel und das ein- resp. umbeschriebene Tetraeder." *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1903, p. 137.

† J. Neuberg: "Ueber die Berührungskugeln eines Tetraeders," *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1907, p. 345.

that an involutory projective relation can be uniquely determined by two arbitrary pairs of corresponding points.

All points, real and imaginary, of a line are represented in a one-to-one way by the real points of a plane; and thence, through stereographic projection (or inversion) by the real points of a sphere. Linear or projective transformations of points on the line are equivalent to those real linear transformations in space of three dimensions which carry the sphere over into itself, and vice versa. *These transformations do not alter the magnitude of any angle between curves on the sphere.* To apply these facts toward proof of the theorem under consideration, let the circumscribed tetraedron have vertices A, B, C, D, and for points of contact, A_1 in the face BCD, B_1 in ACD, etc.

We wish to show, for example, that three angles at A_1 made by rays A_1B, A_1C, and A_1D are equal respectively to three angles formed at B_1 by rays drawn to A, D, and C. All these rays are of course tangent to the sphere.

Determine an involution by the pairs of points A_1 and B_1, C_1 and D_1 It will carry planes into planes, and planes tangent to the sphere into other such planes; in particular the four faces of the tetraedron are permuted among themselves, so that the vertices A and B are exchanged, likewise C and D. Hence the angles on the spherical surface measured by BA_1C, CA_1D, DA_1B are transformed into angles of unchanged magnitude measured respectively by AB_1D, DB_1C, CB_1A, and this proves the theorem.

This same method may be applied to draw new relations from the theorem that a collineation can be found which shall transform three given points of a straight line into three arbitrary points. Upon the sphere let the six edges of a tetraedron be tangent in six points, which may be denoted most clearly by double indices, 12, 13, 14, 34, 42, 23; *e. g.*, the edge AB shall touch the sphere at 12, etc. Through each point of contact pass a plane containing the opposite edge. We shall prove that *these planes form three pairs which intersect the sphere in orthogonal pairs of circles; and that the right angles between these circles at any contact are bisected by the traces of two other planes, each containing four points of contact.*

The relations appear on inspection as true for the regular tetraedron; and to this form every set of tangent edges of an oblique tetraedron can be brought by the admissible transformations. For the three contacts 12, 23, 31 can be brought by

such a transformation into coincidence with the contacts of three base edges of a regular tetraedron. These three points determine then the vertex 4, and with it the other three contacts; not uniquely, but as one of two definite positions: an interchange of two contacts, if necessary, will bring the vertex 4 to coincidence with that of the regular tetraedron. By this collinear transformation all the planes above mentioned come to the positions of the nine correspondingly situated planes in the regular figure, thus proving the theorem.

It is to be noticed that this tetraedron whose edges all touch a sphere is not an arbitrary figure of its class, since it depends on six constants, beside the radius of the sphere and three for the center, — ten in all, while the tetraedron with four vertices free involves twelve constants. Two sufficient conditions for inscriptibility of a sphere in this sense are these, that the sum of the lengths of two opposite edges shall be the same for each of the three opposite pairs. Less symmetrically, we note that circles inscribed in any two lateral faces must be tangent to that inscribed in the base, and from this will follow the same relation in the third lateral face.

The set of six contacts of six edges of a tetrahedron with a sphere divides naturally into threes in four ways; for example, 12, 13, 14; 23, 34, 42, three in one face and three on edges having the opposite vertex in common. *Each set of three may exchange rôles with its complementary set of three*, as is seen by applying to the figure the polar reciprocal transformation with respect to the sphere, rotating each edge through a right angle to the conjugate position. The three planes through four points of contact go into poles, whose relations to the six contacts might invite attention.

Finally it may be recalled that all the constructions suggested by these figures can be carried out in the plane, upon the six contacts of four mutually tangent circles; also that the complementary sets of three points indicate an interesting problem in the covariants of a binary cubic form.

VASSAR COLLEGE,
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SINGULAR POINTS OF A SIMPLE KIND OF DIFFERENTIAL EQUATION OF THE SECOND ORDER.

BY PROFESSOR C. A. NOBLE.

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In a series of four memoirs in the *Journal de Mathématiques* (series 3, numbers 7, 8; series 4, numbers 1, 2) Poincaré has, among other things, discussed the topology of curves defined by ordinary differential equations of a simple character. In a recent course of lectures Hilbert laid considerable stress on the importance of these results and exhibited an elegant method for obtaining them in the case of a differential equation of the form $dy/dx = (cx + dy)/ax + by)$. In the following paper I have shown how the same method can be used for an ordinary differential equation of the second order. My results tally with those of Poincaré insofar as the latter are enumerated; but they are more detailed than his and are, I think, more simply obtained.

Given (1) $d^2y/dx^2 = (dx+ey+f\cdot dy/dx)/(ax+by+c\cdot dy/dx)$, a, b, c, d, e, f real constants. Put (2) $dy/dx = z$ and (1) becomes

$$(3) \qquad dz/dx = (dx + ey + fz)/(ax + by + cz).$$

Now write

$$(4) \quad dx/dt = ax + by + cz, \quad dy/dt = z, \quad dz/dt = dx + ey + fz.$$

Multiply the second and third of these equations by m and n respectively and add,

$$(5) \quad d(x + my + nz)/dt = (a + nd)x + (b + ne)y + (c + m + nf)z.$$

Equate the second member of (5) to $\lambda(x + my + nz)$ and determine m, n and λ accordingly. We find

$$(6, 7) \qquad n = (\lambda - a)/\lambda, \quad m = (bd - ac + e\lambda)/d\lambda,$$

$$(8) \qquad \lambda^3 - (a + f)\lambda^2 + (af - cd - e)\lambda - (bd - ae) = 0.$$

Indicate the three values of λ arising from (8) and the corresponding values of m, n by subscripts. Put

9) $x + m_1 y + n_1 z = \xi, \quad x + m_2 y + n_2 z = \eta, \quad x + m_3 y + n_3 z = \zeta$

and we obtain as the equivalent of (4)

(10) $\qquad d\xi/dt = \lambda_1 \xi, \quad d\eta/dt = \lambda_2 \eta, \quad d\zeta/dt = \lambda_3 \zeta$

or, eliminating t,

(11) $d\eta/d\xi = \kappa \cdot \eta/\xi, \; d\zeta/d\xi = \mu \cdot \zeta/\xi,$ where $\kappa = \lambda_2/\lambda_1, \; \mu = \lambda_3/\lambda_1.$

The problem in hand is to investigate the nature of the critical point (0, 0, 0) in x, y, z space for the equations (4). By virtue of the substitutions (9) this point goes over into (0, 0, 0) in ξ, η, ζ space. There are three principal cases to consider, depending upon the roots of (8). They are: I. The λ_i all real and different; II. Two λ_i conjugate complex; III. Two or more λ_i identical.

I. $\lambda_1, \lambda_2, \lambda_3$ real and different. The integrals of (11) are

(12, 13) $\qquad \eta = C\xi^{\kappa}, \quad \zeta = D\xi^{\mu} \quad (C, D \text{ integration constants}).$

The intersections of these two surfaces constitute ∞^2 integral curves. How are these curves distributed in the immediate vicinity of (0, 0, 0)? How many pass through (0, 0, 0)? An answer to these questions amounts to a characterisation of the critical point. There are three possibilities. They arise according as κ, μ are (i) both > 0, (ii) both < 0, or (iii) unlike in sign.

(i) $\kappa > 0, \mu > 0$. By appropriately naming the roots of (8) we can make $\kappa > 1, \mu > 1$. Every cylinder (12) passes through the ζ-axis and is tangent to the plane $\eta = 0$. Every cylinder (13) passes through the η-axis and is tangent to the plane $\zeta = 0$. There are therefore ∞^2 integral curves passing through (0, 0, 0), all tangent to one of their number, viz., to the ξ-axis. Hilbert would probably call this singular point a Scheitel-scheitelpunkt.

(ii) $\kappa < 0, \mu < 0$. We may write (12), (13) in the form

(12, 13) $\qquad \eta\xi^{-\kappa} = C, \quad \zeta\xi^{-\mu} = D.$

The only integral curves passing through (0, 0, 0) now are $\xi = 0 = \eta$, $\eta = 0 = \zeta$, and $\zeta = 0 = \xi$. (12) represents a family of hyperbolic cylinders with generators parallel to the

ζ-axis; (13) a family of hyperbolic cylinders with generators parallel to the η-axis. All the ∞^2 integral curves, barring the three above mentioned, tend to hug the ξ-axis and the $\eta\zeta$-plane. Hilbert would call this singular point a Sattel-sattelpunkt.

(iii) $\kappa > 0$, $\mu < 0$ (or $\kappa < 0$, $\mu > 0$). Write (12), (13) in the form

$$\eta = C\xi^{\kappa}, \quad \zeta\xi^{-\mu} = D. \tag{12, 13}$$

Every cylinder (12) yields two integral curves through (0, 0, 0), viz., the curves of intersection with $\xi = 0$ and with $\zeta = 0$ (these two planes are members of (13)). There are thus ∞^1 integral curves through the origin, all of them lying in one plane, except one, this one being the ζ-axis. Hilbert would probably call this singular point a Scheitel-sattelpunkt mit isoliertem Strahl.

II. λ_1 real; λ_2, λ_3 conjugate complex. The integrals of (10) may now be written

$$\eta = C\xi^{\kappa}, \quad \zeta = D\eta^{\nu} \tag{12, 14}$$

$$(\kappa = \lambda_2/\lambda_1,\ \nu = \lambda_3/\lambda_2,\ C,\ D \text{ arbitrary constants}).$$

If we seek an interpretation of (12), (14) in real space by means of the substitutions

$$\begin{aligned} \xi &= x + m_1 y + n_1 z = X, \\ \eta &= x + m_2 y + n_2 z = x + m_2' y + n_2' z + i(m_2'' y + n_2'' z) = Y + iZ, \\ \zeta &= x + m_3 y + n_3 z = x + m_2' y + n_2' z - i(m_2'' y + n_2'' z) = Y - iZ, \end{aligned} \tag{15}$$

whereby

$$m_2 = m_2' + im_2'', \quad m_3 = m_2' - im_2'', \quad n_2 = n_2' + in_2'', \quad n_3 = n_2' - in_2'',$$

we get from (12)

$$\begin{aligned} Y + iZ &= CX^{\lambda_2/\lambda_1} = CX^{\lambda_2'/\lambda_1} \cdot e^{i(\lambda_2''/\lambda_1)\log X} \\ &= CX^{\lambda_2'/\lambda_1}\{\cos[(\lambda_2''/\lambda_1)\log X] + i\sin[(\lambda_2''/\lambda_1)\log X]\} \\ &\qquad (\lambda_2 = \lambda_2' + i\lambda_2''). \end{aligned} \tag{16}$$

Separating real and imaginary parts in (16) and equating,

$$Y = CX^{\lambda_2'/\lambda_1} \cos [(\lambda_2''/\lambda_1) \log X], \tag{17}$$

$$Z = CX^{\lambda_2'/\lambda_1} \sin [(\lambda_2''/\lambda_1) \log X]. \tag{18}$$

Squaring (17), (18) and adding we obtain

$$Y^2 + Z^2 = C^2 X^{2\lambda_2'/\lambda_1}, \tag{19}$$

a surface, or rather ∞^1 surfaces, of revolution upon which all the ∞^2 integral curves lie. Again, we have from (14) by means of (15),

$$Y - iZ = D(Y + iZ)^\nu. \tag{20}$$

Let us introduce polar coördinates into the Y, Z plane by putting $Y + iZ = \rho e^{i\phi}$. We obtain from (20)

$$\begin{aligned} \rho &= \text{constant} \cdot e^{i\phi(1+\nu)/(1-\nu)} \\ &= \text{constant} \cdot e^{\phi\lambda_2'/\lambda_2''} = e^{(\lambda_2'/\lambda_2'')(\phi + \text{const.})}, \end{aligned} \tag{21}$$

a family of ∞^1 spiral cylinders winding about the X-axis. The ∞^2 integral curves are thus the intersections of the two sets of surfaces in X, Y, Z space

$$Z = CX^{\lambda_2'/\lambda_1} \cdot \sin [(\lambda_2''/\lambda_1) \log X], \tag{18}$$

$$\rho = e^{(\lambda_2'/\lambda_2'')(\phi + C')} \quad (C' \text{ an arbitrary constant}). \tag{21}$$

The critical point (0, 0, 0) in X, Y, Z space, which is likewise the origin in x, y, z space, might be called a Strudelpunkt, by analogy with the corresponding situation which arises for a differential equation of the first order. The surfaces (18) lie entirely in space $X > 0$, and they all contain the Y-axis as a "stop line" when $\lambda_2'/\lambda_1 > 0$. If $\lambda_2'/\lambda_1 \not> 0$, none of the surfaces (18) (except the one for $C = 0$) contains the Y-axis—they cease being defined when X approaches zero. In fact, these cylinders (18) have, along the Y-axis, a definite tangent plane only when $\lambda_2'/\lambda_1 > 1$. In the present case, therefore, there is no integral curve containing (0, 0, 0). If, however, $\lambda_2'/\lambda_1 > 1$, there will be ∞^2 curves which approach (0, 0, 0) asymptotically in space $X > 0$.

For the topological discussion of the equation of the first order there is but one integral equation, viz., the analogon of

(21), and the special assumption $\lambda_2'/\lambda_2'' = 0$ then yields the special class of singular point called Wirbelpunkt, which is completely enclosed by each one of the ∞^1 integral curves $\rho =$ constant. Here, however, in space, no such situation can arise, because (18) loses geometric meaning for $\lambda_2'/\lambda_2'' = 0$. There appears to be nothing in the present problem analogous to the Wirbelpunkt in the plane.

III_1. $\lambda_3 = \lambda_2 \neq \lambda_1$. The three equations (10) now shrink to two, and another means of integrating (4) must be devised. Take the three equations

$$d\xi/dt = \lambda_1\xi, \quad d\eta/dt = \lambda_2\eta, \quad dz/dt = dx + ey + fz.$$

Assume the second member of the third equation identically equal to $\alpha\xi + \beta\eta + \gamma z$. This necessitates

$$\alpha = \lambda_1 d/(\lambda_1 - \lambda_2), \quad \beta = -\lambda_2 d/(\lambda_1 - \lambda_2), \quad \gamma = \lambda_2,$$

so that, instead of (10), we have as our three equations

$$\begin{aligned} &d\xi/dt = \lambda_1\xi, \quad d\eta/dt = \lambda_2\eta, \\ &dz/dt = (\lambda_1 d/(\lambda_1 - \lambda_2))\xi - (\lambda_2 d/(\lambda_1 - \lambda_2))\eta + \lambda_2 z. \end{aligned} \tag{22}$$

Eliminating t,

$$\begin{aligned} &d\xi/d\eta = (\lambda_1/\lambda_2)(\xi/\eta), \\ &dz/d\eta = (\lambda_1 d/\lambda_2(\lambda_1 - \lambda_2))(\xi/\eta) - d/(\lambda_1 - \lambda_2) + z/\eta. \end{aligned} \tag{23}$$

Solving the first of these and making use of the result in the second we obtain as complete integral

$$\xi = C\eta^{1/\kappa}, \tag{12}$$

$$z = (C\lambda_1 d/(\lambda_1 - \lambda_2)^2)\eta^{1/\kappa} - (d/(\lambda_1 - \lambda_2))\eta \log \eta + D\eta \tag{24}$$

(C, D arbitrary).

As before noted, every surface (12) passes through the origin in ξ, η, z space, which is also the origin in x, y, z space. Different possibilities can arise according as κ is > 0 or < 0, and according as d is zero or not.

$d \neq 0$, $\kappa > 0$. Each of the surfaces (24) contains the ξ-axis as a "stop line," since η must be $\geqq 0$; and each of these surfaces is tangent along the ξ-axis to the ξz-plane, since

$dz/d\eta = \infty$ for $\eta = 0$. Hence all the ∞^2 integral curves enter (0, 0, 0) in the half space $\eta \geqq 0$. This singular point is of the sort which Hilbert calls Endpunkt.

$d \neq 0$, $\kappa < 0$. Write (12), (24) in the following form

$$\xi\eta^{-1/\kappa} = C, \tag{12}$$

$$z\eta^{-1/\kappa} = C\lambda_1 d/(\lambda_1-\lambda_2)^2 - (d/(\lambda_1-\lambda_2))\eta^{1-1/\kappa}\cdot\log\eta + D\eta^{1-1/\kappa}. \tag{24}$$

For $C = 0$ (12) gives the planes $\xi = 0$, $\eta = 0$, while (24) gives ∞^1 cylinders in space $\eta \geqq 0$ and having the ξ-axis as "stop line." The intersections of (12) and (24) therefore furnish, for $C = 0$, ∞^1 curves containing (0, 0, 0), all but one of them lying in the half-plane $\xi = 0$ $\eta \geqq 0$ and having (0, 0, 0) as "stop point," while that exceptional one is the ξ-axis itself.

For $C \neq 0$, (12) gives a set of ∞^1 hyperbolic cylinders with generators parallel to the z-axis, while (24) gives (for η small but $\not< 0$) a set of ∞^2 hyperbolic half-cylinders with generators parallel to the ξ-axis. The intersections of (12) and (24), therefore, furnish, for $C \neq 0$, ∞^2 curves all lying in space $\eta > 0$; and none of these contains (0, 0, 0). The singular point for $d \neq 0$, $\kappa < 0$ is thus of a kind distinct from those previously considered. Only one integral curve passes through (0, 0, 0); ∞^1 integral curves have (0, 0, 0) as "stop point;" and ∞^2 integral curves pass by (0, 0, 0) in arbitrary proximity. This singular point might be call Sattel-Endpunkt mit isoliertem Strahl.

$d = 0$, $\kappa \neq 0$. Our integrals now take the form

$$\eta = C\xi^{\kappa}, \quad z = D\eta. \tag{12, 25}$$

We have here no new kind of singular point. If $\kappa > 0$, the origin is a Scheitelpunkt. If $\kappa < 0$, the origin is a Sattelpunkt.

$\mathrm{III}_{\mathrm{II}}$. $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$. The three equations (10) now shrink to one, and we must seek still another means of integrating (4). Assume

$$dx/dt = \alpha x + \beta\eta + \gamma z, \quad d\eta/dt = \lambda\eta, \quad dz/dt = \alpha' x + \beta'\eta + \gamma' z, \tag{26}$$

where α, β, γ, α', β', γ' are to be determined so that the equations (26) shall be identities in x, y, z. The appropriate values of these constants are

$$\alpha=\lambda^2(a-\lambda)/(\lambda^2+e), \quad \beta=bd/(\lambda^2+e),$$
$$\gamma=(e\lambda^2-b\lambda+ab+ce)/(\lambda^2+e),$$
$$\alpha'=\lambda^2 d/(\lambda^2+e), \quad \beta'=ed/(\lambda^2+e),$$
$$\gamma'=\lambda(f\lambda+2e)/(\lambda^2+e).$$

We may now write (26) in the form

$$\begin{aligned} dx/d\eta &= \alpha x/\lambda\eta+\gamma z/\lambda\eta+\beta/\lambda, \\ dz/d\eta &= \alpha' x/\lambda\eta+\gamma' z/\lambda\eta+\beta'/\lambda. \end{aligned} \tag{27}$$

From this we can pass, by the substitution $u=x+(\lambda-\alpha)z/\alpha'$, to the equivalent pair

$$\begin{aligned} du/d\eta &= u/\eta+(\lambda^2+e)/\lambda^2, \\ dz/d\eta &= \lambda du/(\lambda^2+e)\eta+z/\eta+ed/\lambda(\lambda^2+e). \end{aligned} \tag{28}$$

The integrals of (28) are

$$u=\eta\log(C\eta^{1+e/\lambda^2}), \tag{29}$$

$$z=(d/\lambda)[(e+\lambda^2\log C)\eta\log\eta/(e+\lambda^2)+\tfrac{1}{2}\eta(\log\eta)^2]+D\eta. \tag{30}$$

By all the substitutions above used, the origin has remained unchanged, so that in u, η, z space the critical point is still $(0, 0, 0)$. In (29) η may take negative as well as positive values, but not for the same value of C; so that the z axis is a "stop line" for every surface represented by (29). In (30), however, η must be $\gtreqless 0$ as long as $d \neq 0$. We must therefore distinguish two possibilities, according as d is, or is not, zero.

$d \neq 0$. All the surfaces (29) will enter the z-axis tangent to the plane $\eta=0$ except when $1+e/\lambda^2=0$; but this exception may be excluded, since the above transformations would then be impossible. Every surface (30) contains the u-axis as a "stop line" and enters that line tangent to the plane $\eta=0$. The peculiarity of the singular point $(0, 0, 0)$ in the present case is that every curve of intersection of (29) and (30) not only enters it as a "stop point," but enters it from the same octant. The point might still be called an Endpunkt.

$d=0$. The surfaces (30) are now a one-parameter set of planes through the u-axis. All the integral curves have $(0, 0, 0)$ as a "stop point," which is therefore to be classed as Endpunkt.

ZÜRICH, *August* 14, 1907.

THE THEORY OF ELECTRICITY.

Theorie der Elektrizität. Von M. ABRAHAM. Zweiter Band: *Elektromagnetische Theorie der Strahlung.* Leipzig, B. G. Teubner, 1905. x + 404 pp.

THE first volume of Abraham's treatise on electricity, which was written as a revision of Föppl's earlier book on the same subject, has already been reviewed in the BULLETIN.* At the close of that notice the statement was made that "The second volume will be awaited with impatience." The volume appeared within a few months, and by this time the author would probably feel justified in concluding that however much the reviewer may have been impatient for the appearance of the volume, he was not particularly impatient to review it. This very tardiness will, however, be of no inconsiderable aid in writing a comment on the book.

Abraham's second volume deals with the theory of electrons. It is divided into two parts of which the first treats the field and motion of individual electrons and the second the electromagnetic phenomena in ponderable bodies. Although physical science has taken long strides in the past forty months along the path of electron theory, so that now the electron and its major properties must be considered by all as firmly intrenched facts of physics instead of grudgingly acknowledged theories, and although J. J. Thomson in a recent number of the *Philosophical Magazine* has contributed much in the way of enlightenment to our hitherto very vague notion of the nature of positive electricity, nevertheless comparatively little of that which has been accomplished in addition to what was known at the time of the publication of this volume can as yet be construed as offering very material aid in constructing or revising the mathematical theory of electricity from the electronic point of view. What the electron does in the large — a large which may be measured in small fractions of a $\mu\mu$ — is tolerably well known; but what its characteristics and behavior may be within a distance of one or two 10^{13}ths of a centimeter from its center is still much of an eleusinian mystery.

* Volume 11, pp. 383–387 (April, 1905). The date at the end of the review should be February, 1905, not February, 1904. — We may note that a new edition of this first volume has just appeared.

The uninitiated may not be aware that there are at least four distinct electrons which have reached a considerable mathematical development. The first of these may be called the Larmor electron, inasmuch as the best exposition of its properties is found in Larmor's Aether and Matter. It is a mathematical point endowed with a finite charge of electricity which creates a certain strain or "beknottedness" in the surrounding ether. Of course nobody, Larmor least of all, would maintain that the real physical electron was thus devoid of extension in space. The conception is merely preliminary and is adopted for the purpose of obtaining mathematical simplicity. The second electron may be called the Abraham electron. It is discussed in detail in the book under review and previously to the publication of this book it had already received great development at the hands of the author. Its dimensions are finite and its configuration is spherical with a radius r which probably lies within the limits $10^{-13} < r < 2 \cdot 10^{-13}$. It is rigid. As Lorentz is so nearly the father of all mathematical electron theory, it is perhaps unfair to designate any particular electron by his name. The particular electron, however, which is called the Lorentz electron is also of finite dimensions. When at rest it is spherical, but when animated with a motion of translation it becomes an ellipsoid of revolution with the axis of revolution shortest and directed along the line of motion. If r be the radius of the sphere at rest and if β be the ratio of the velocity of the electron to the velocity of light, the semiaxes of the ellipsoid perpendicular to the line of motion remain equal to r, whereas the axis along the path becomes $r(1-\beta^2)^{\frac{1}{2}}$. This introduces a shrinking along the line of motion and was adopted for the purpose of explaining the troublesome Michelson-Morley experiment. Finally there is the nearly simultaneous creation of Bucherer and Langevin — an electron which shortens in the direction of motion but expands in the perpendicular direction by an amount sufficient to make the volume of the ellipsoid equal to the volume of the original sphere; the new semiaxes are $r(1-\beta^2)^{-\frac{1}{2}}$, $r(1-\beta^2)^{-\frac{1}{2}}$, and $r(1-\beta^2)^{\frac{1}{2}}$.

Fortunately the greater part of the mathematical theory of each of these four electrons is common to the other three. It is only in effects which may be called of the second order that radical differences occur, and even then the differences are not always great. Two very delicate experiments, one the Michelson-Morley experiment already referred to and the other the

Kaufmann experiment, have perhaps been the chief tests of the applicability of the different theories. On the hypothesis that an electron is a charged particle of finite size, the particle in motion will possess an inertia of electric origin which will increase with the velocity and become infinite when the velocity approaches that of light. The increase is very slow at first and scarcely becomes appreciable until the fraction β has passed the the value 0.5. If it be assumed that the entire inertia of the electron is of electric origin — a fact which seems to be clearly indicated by Kaufmann's earlier experiment — it follows that the mass of the electron, whether the transversal or the longitudinal mass, is a subject for purely electromagnetic experiment and theory.

If m denote the transversal mass and m_0 be the value of this mass when $\beta = 0$, the mass for any velocity may be written as $m_0\Phi(\beta)$, where $\Phi(\beta)$ is a function of β which has different expressions according to the different theories, namely,

$$\Phi(\beta) = \frac{3}{4}\frac{1}{\beta^2}\left(\frac{1+\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta} - 1\right) \text{ for Abraham's electron,}$$

$$\Phi(\beta) = (1-\beta^2)^{-1} \text{ for Lorentz's electron,}$$

$$\Phi(\beta) = (1-\beta^2)^{-1} \text{ for Bucherer's electron.}$$

Since the appearance of Abraham's book, Kaufmann has carried out an extended series of experiments * from which he infers that either the Abraham or the Bucherer electron represents the experimental facts within an error which is less than the experimental error, whereas Lorentz's electron gives calculated values which depart from the observed values by an amount considerably in excess of the experimental errors. Hence, so far as these particular experiments are concerned, it appears that there are still two types of electron to dispute the field.

Another lengthy investigation † which has appeared since Abraham's book is due to Poincaré. Instead of being experimental it is purely mathematical and based on the Michelson-Morley experiment and the principle of relativity. Here the

* Printed in the *Annalen der Physik*, volume 19 (1906). The range of values of β is about $0.55 < \beta < 0.97$. It is really remarkable that two theories should agree within the small error of two or three per cent. over such a wide range.

† Printed in the *Rendiconti di Palermo*, volume 21 (1906).

theory of groups of transformations does valiant service. The principle of relativity is the postulate that there never can be obtained a method for distinguishing absolute motion; and this certainly accords with past experiments on the subject, which have constantly afforded merely negative results. Poincaré examines the consequences of the assumptions of Abraham, Lorentz, and Bucherer. He finds, among other things, that if an electron is constrained by a relation between its three axes and if no other forces not of electromagnetic origin act upon the electron, then the Bucherer-Langevin hypothesis is the only one admissible. On the other hand this hypothesis does not admit the principle of relativity. Lorentz had previously established this result in a different manner. Poincaré finds also that the only hypothesis which is in accord with the principle of relativity is that of Lorentz; but here it is necessary to introduce forces of order other than electromagnetic, and these forces are derivable from a potential which is proportional to the volume of the electron.

It is needless to remark that in a subject so replete with hypotheses and difficulties as the electron theory there is always a possibility that something has been overlooked and that some day another interpretation may be available which will vitiate present conclusions: but Kaufmann is a particularly careful and habile experimenter and Poincaré is no less an astute mathematician, and it is interesting to combine their conclusions even if to-morrow may call for a revision of the inference. From Kaufmann's experiment let us conclude to throw out the Lorentz electron. Poincaré's work then shows that the principle of relativity cannot subsist. It is certainly more satisfactory philosophically and scientifically to be left with the hope that some time we may be able to distinguish absolute motion than to feel that we shall in nowise be able to do so. Again making an appeal to Poincaré's investigation, we may infer that the Bucherer-Langevin electron is the best to adopt inasmuch as it does not necessitate the introduction of other forces than those of electromagnetic origin and those involved in the relation between the axes of the ellipsoid of revolution. Surely students of electricity will prefer the simple expression of $\Phi(\beta)$ in this system to that given by Abraham, and to any follower of Maxwell the supposition that the volume of the electron is constant must be a source of considerable consolation.

The first chapter of Abraham's book is on the physical and mathematical foundation of the electron theory. The treatment is excellent. The historical notes, the collection of evidence culminating in the irresistible inference of the existence of electrons and their simple relations to problems of electromagnetism, the insistence on the numerical values of the fundamental magnitudes, and the systematic classification of radiation make greatly for the ease and contentment of the reader. No sooner are the fundamental equations set up than the author proceeds to develop the ideas of electromagnetic momentum and moment of momentum. This parallel to ordinary mechanics aids in the comprehension of the text and allows the author to emphasize the differences between the laws of electricity and those of matter. The question of the inequality of action and reaction, which at first caused considerable difficulty to some investigators, is treated with a detail and clearness which leave nothing to be desired. The important results are stated here and throughout the book in spaced type so as to catch the eye. The integration of the equations to obtain the formulas for the potentials is accomplished by a method due to Abraham and based on the familiar method given for a special case in Weber's Partielle Differentialgleichungen. This is the only complicated piece of analysis in the chapter.

The second chapter treats the radiation of waves from a point charge in motion and is consequently of importance in the theory of optics. It will not be amiss to call especial attention to the words point charge. The author takes pains to indicate what portions of his theory are independent of his particular assumption of rigid spherical electrons. This is of high value to the reader, for whom it is a matter of importance to have those portions of the work which would be true on any of the current hypotheses concerning electrons separated from the consequences due to Abraham's special electron. This chapter, then, is concerned with what happens at a considerable distance from the electron. The model for a radiating source of light is the electric dipole, consisting of a fixed positive charge and a moving negative charge. The loss of energy through radiation is computed, and not merely computed as a formula but actually turned into numbers from available experimental data. The emphasis laid upon the order of numerical magnitude of the quantities involved in a calculation will almost inevitably serve to differentiate a true mathematical physicist from a mathema-

tician who merely looks to physics for problems to which to apply his analysis. In this chapter the Zeeman effect naturally comes in for a detailed discussion. The field due to an electron moving uniformly or with an acceleration is taken up and applied to the theory of a moving source of light. The questions of longitudinal and transverse vibrations are handled separately, and the matter of the reaction of the radiation on the source is not overlooked. It is a happy idea of the author's not to be in any haste in passing over these fundamental points.

The mechanics of electrons is the subject of the third chapter. There is an introduction in which the reasons for assuming that the mass of the electron is wholly of electromagnetic origin are outlined and in which the relation of the electromagnetic cosmos to the mechanical cosmos discussed by Hertz in his Mechanics is set forth. The author then gives a detailed argument in favor of his rigid spherical electron. His aim is to construct the world on a purely electromagnetic basis, and the assumption of any other hypothesis concerning the electron appears to him to necessitate the introduction of enormous elastic forces of non-electromagnetic origin. This seems scarcely conclusive, especially in view of Poincaré's recent memoir. It is well known that when a charged sphere is in motion the lines of force, alias the Faraday tubes, are drawn toward a plane through the center of the sphere and perpendicular to the path. Why this might not quite naturally result in the Bucherer-Langevin deformation is hard to see. In fact it would seem that when an electron is in motion it would require enormous forces of rigidity which were not of electromagnetic origin to preserve the spherical shape of the electron. If the author were to reprint his book now, he would doubtless give this question a more thorough treatment.

In this chapter on the mechanics of electrons Abraham again keeps close enough to ordinary mechanics to speak of momentum, moment of momentum, moment of inertia, and the lagrangian function. This part of the work is as interesting as any and is highly to be recommended for its pedagogic excellence, especially in view of the fact that when the book was written it had many persons to convince as well as to instruct. The matter of electromagnetic mass, whether transversal or longitudinal, is expounded in all detail and is set into relation with the lagrangian function. The Lorentz electron is treated and its relation to the author's is indicated. The discussion ends with the statement that Lorentz had shown that his electron satisfied

the (earlier) Kaufmann experiments about as well as Abraham's and with the expression of the hope that further refinements of the experiment would serve to differentiate between the two. That hope was soon to be realized in favor of the author. Unfortunately no word is mentioned concerning Bucherer's electron. This is a serious omission, although perhaps unavoidable, inasmuch as Bucherer's book * had appeared only some six or eight months before Abraham's. The chapter concludes with some interesting though partly speculative investigations such as the discontinous motion of electrons (useful in the explanation of Röntgen rays), the force in the interior of an electron, and the uniform motion of an electron with a velocity greater than that of light.

The author then comes to the second part of his work, that on electromagnetic phenomena in ponderable bodies. This he divides into two chapters, the first on stationary bodies, the second on moving bodies. After a preliminary discussion of what is to be understood by the term physically small, he introduces the method of averages to derive the ordinary equations of the electromagnetic field in bodies at rest. The dispersion of electromagnetic waves and the connection with the index of refraction, the magnetic rotation of the plane of polarization, the question of magnetization, and electric conduction in metals are treated from the point of view of electron theory. The name of Drude occurs frequently in these pages, but we do not find that of J. J. Thomson. To be sure, a great deal of Thomson's work † in this particular direction had not appeared at the time of writing and considerable more of it might have been thought to be still in too speculative a state and insufficiently capable of presentation in a form assimilable with the rest of the book; this, however, was hardly true of all of it. Other omissions, such as the theory of optical rotation in crystals and solutions and the explanation of the Peltier effect, may be explained by the simple statement that even now electrons have not enabled us to account for all electrical phenomena. As a general criticism, though not a severe one, we may say

* Mathematische Einführung in die Elektronentheorie. B. G. Teubner, 1904, 148 pp. This is perhaps the best short account of an electron theory. On account of its brevity it is more exclusively mathematical and consequently of greater average difficulty than the work under review.

† We may mention his book Electricity and Matter, Charles Scribner's Sons, 1904; 162 pp. Also the extremely recent Corpuscular Theory of Matter, Constable, 1907; 180 pp. The latter we have not yet seen.

that it would have added to the interest of the book if the author had gone more into detail from time to time as regards the things which were yet to be accounted for by the theory instead of confining himself so exclusively to matters which he actually was in a position to analyze. In conclusion to the chapter there is a lengthy discussion of the mathematics of wireless telegraphy.

The concluding chapter on moving bodies commences with the careful and critical derivation of the equations of the field followed by a discussion of Fizeau's experiment. The crucial nature of this experiment for deciding between Lorentz's optics and Hertz's is brought out. There follow sections on the pressure of radiation on surfaces in motion, whether they be black or reflecting. It would have been possible to add somewhat to the interest of this question by entering upon Poynting's applications of the results to cosmical speculations. In the section on the temperature of radiation we find the laws of Kirchhoff and Wien. Here the author, as in so many other places, goes into the matter numerically. Next follows the treatment of the Michelson-Morley experiment and its crucial evidence against a stagnant ether or in favor of a contraction along the path. This leads to a presentation of the optics of Lorentz and Cohn; and with that the book comes to a close except for the extensive index.

From what has been said it cannot fail to appear that we have in this treatise a work which deals with the most fundamental questions of physics and sets them into relation with the latest developments of theory and experiment. In only a few places is the analysis complicated, and everywhere there is an abundance of physical data which are frequently worked out to their numerical consequences. To an unusually large extent the book represents the work of individual investigation on the part of the author. It could not have been written as a compilation from the accomplishments of others. Whether the Abraham electron shall persist or be cast aside, the greater portion of the present volume will remain, and most of the rest will have to be changed but little. Clearly those who impatiently awaited the appearance of this second volume have not been disappointed in their expectations of it.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *December*, 1907.

NOTES.

THE opening (January) number of volume 9 of the *Transactions of the* AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Certain periodic orbits of k finite bodies revolving about a relatively large central mass," by F. L. GRIFFIN; "Further note on Maclaurin's spheroid," by G. H. DARWIN; "Potential functions on the boundary of their regions of definition" and "Double distributions and the Dirichlet problem," by O. D. KELLOGG; "Groups defined by the orders of two generators and the order of their commutator," by G. A. MILLER; "Projective differential geometry of curved surfaces" (second memoir), by E. J. WILCZYNSKI.

THE opening (January) number of volume 30 of the *American Journal of Mathematics* contains the following papers: "Plane quintic curves which possess a group of linear transformations" and "On birational transformations of curves of high genus," by V. SNYDER; "Surfaces with the same spherical representation of their lines of curvature as spherical surfaces," by L. P. EISENHART; "On the factoring of composite hypercomplex number systems," by H. B. LEONARD; "A new method in geometry," by E. LASKER; "Groups generated by n operators each of which is the product of $n-1$ remaining ones," by G. A. MILLER.

AT the meeting of the London mathematical society held on December 12 the following papers were read: By S. T. SHOVELTON, "A formula in finite differences and its application to mechanical quadrature"; by A. E. H. LOVE, "Weierstrass's excess-function in the calculus of variations."

AT the Dresden meeting of the Deutsche Mathematiker Vereinigung Professor FELIX KLEIN was elected president for the current year, and also named official representative of the society at the fourth international congress of mathematicians, to be held in Rome in April. The present volume (17) of the *Jahresbericht* will have different pagination for the memoirs, from the part devoted to communications, notes, etc. A committee consisting of Mr. ACKERMANN-TEUBNER and Professor GUTZMER was appointed to consider ways and means of making

new mathematical books published outside of Germany more accessible. A committee was appointed to cooperate with the Swiss committee to consider the publication of the works of Euler. The committee on methods of instruction will be augmented by inviting two representatives from the various academies and societies that wish to cooperate in the movement of reorganizing mathematical instruction in the secondary schools. The next meeting of the Vereinigung will be held in Cologne in September 1908. It now numbers 703 members.

AT the annual public meeting of the Paris academy of sciences, held on December 16, 1907, the following prizes were awarded for memoirs in pure and applied mathematics: Francoeur prize (fr. 1000) to E. LEMOINE, for his contributions to geometry; Bordin prize (fr. 3000) divided between F. ENRIQUES and F. SEVERI, for the solution of the problem announced in the BULLETIN, volume 11, page 338; Vaillant prize (fr. 4000) divided among J. HADAMARD, A. KORN, G. LAURACELLA and T. BOGGIO, for contributions to the theory of the equilibrium of supported elastic plates; Montyon prize (fr. 700) to M. CUENOT, for his analysis of the flexibility of railroad rails; Poncelet prize (fr. 2000) to Col. RENARD for his researches in mechanics; Lalande prize (fr. 540) to T. LEWIS, for his contributions to astronomy; Valz prize (fr. 460) to M. GIACOBINI for his work in astronomy; Pontécoulant prize (fr. 700) to M. GAILLOT for celestial mechanics; Petit d'Ormay prize (fr. 1000) to P. DUHEM for his work in physical chemistry; Laplace prize (complete works of Laplace) to L. DAUM for his work in mechanics.

The following university courses in mathematics are announced:

OXFORD UNIVERSITY (*Hilary Term*). — By Professor W. ESSON: Comparison of analytic and synthetic methods in the theory of conics, two hours; Synthetic geometry of cubics, one hour. — By Professor E. B. ELLIOT: Elements of elliptic functions, two hours; Theory of numbers, II, one hour. — By Professor H. H. TURNER: Elementary mathematical astronomy, two hours. — By Professor A. E. H. LOVE: Theory of the potential, two hours; Elements of the calculus, two hours. — By Mr. P. J. KIRKBY; Higher plane curves, two hours. — By Mr. A. L. DIXON: Calculus of finite differences, one hour.

— By Mr. J. E. CAMPBELL: Differential equations, II., two hours. — By Mr. C. H. THOMPSON: Dynamics of a particle, two hours. — By Mr. H. T. GERRANS: Hydrodynamics, two hours — By Mr. C. E. HASELFOOT: Geometric optics, two hours. — By Mr. J. W. RUSSELL: Determinants, two hours. — By Mr. A. L. PEDDER: Trigonometry, one hour. — By Mr. A. E. JOLLIFFE: Analytic geometry, II, two hours. — By Mr. R. F. MCNEILE: Integral calculus, two hours. — By Mr. E. H. HAYES: Mechanics, three hours. — By Mr. C. H. SAMPSON: Solid geometry, two hours.

THE Royal Belgian academy of sciences has awarded a prize of 800 francs (prize in geometry for 1907) to Professor G. BORDIGA, of the University of Padua; it also awarded a prize of 600 francs to Professor U. PERAZZO, of the military academy of Turin.

PROFESSOR FELIX KLEIN, of the University of Göttingen, has been appointed representative of the Universities in the upper house of the German parliament.

PROFESSOR D. HILBERT, of the University of Göttingen, has been decorated with the Maximilian order of Bavaria.

DR. A. H. BUCHERER, of the University of Bonn, has been promoted to an associate professorship of mathematics.

DR. P. MUTH, although not connected with any institution of learning, has received the title of professor, in recognition of his mathematical writings.

DR. P. BURGATTI, of the University of Rome, has been appointed associate professor of theoretical mechanics at the University of Messina.

PROFESSOR E. ALMANSI, of the University of Pavia, has been promoted to a full professorship.

PROFESSOR R. MARCOLONGO, of the University of Messina, has been appointed professor of theoretic mechanics at the University of Naples.

DR. G. PICCIATI, of the University of Padua, has been appointed associate professor of mathematical physics at the University of Bologna.

PROFESSOR O. TEDONE, of the University of Genoa, has been promoted to a full professorship of mechanics.

PROFESSOR G. VIVANTI, of the University of Messina, has been appointed professor of the calculus at the University of Pavia.

DR. E. BIANCHI has been appointed docent in mathematical astronomy at the University of Rome.

DR. L. SINIGALLIA has been appointed docent in the calculus at the University of Pavia.

DR. A. DENIZOT has been appointed docent in mathematics at the technical school of Lemberg.

DR. F. BERNSTEIN, of the University of Halle, has been appointed associate professor of mathematics, with particular reference to insurance, at the University of Göttingen.

DR. F. GRAEFE, associate professor of mathematics at the technical school of Darmstadt, has been given the title of honorary professor of mathematics.

PROFESSOR G. KOWALEWSKI, of the University of Bonn, has declined the call to the mining academy at Clausthal.

DR. PFLÜGER, of the University of Bonn, has been appointed associate professor of mathematical physics.

DR. M. WINKELMANN has been appointed docent in mathematics at the technical school of Karlsruhe.

PROFESSOR G. BLAZEK, of the technical school at Prague, has retired.

DR. H. F. BAKER has been appointed chairman of the examiners of the mathematical tripos, part II, for 1908.

PROFESSOR SIMON NEWCOMB has been elected foreign member of the academy of sciences at Göttingen.

DR. G. W. HILL has been elected honorary member of the Royal society of Edinburgh.

PROFESSOR E. O. LOVETT, of Princeton University, has resigned his professorship to accept the presidency of the Rice Institute, Houston, Texas.

MR. G. E. RAMSDELL has been appointed professor of mathematics at Bates College, Lewiston, Maine, successor to the late Professor J. H. RAND, who died November 7, 1907.

DR. L. COHEN has been appointed assistant professor of mathematics at the George Washington University.

SIR WILLIAM THOMSON (Lord Kelvin) died at Glasgow, December 17, 1907. He was born at Belfast, June 25, 1824, and graduated from St. Peter's, Cambridge, in 1845, being first Smith's prizeman and second wrangler as well as oarsman in the victorious Cambridge crew. The next year he was appointed professor of natural philosophy at the University of Glasgow, which post he held for over 50 years. His scientific assistance made the Atlantic cable a success; for this achievement he was knighted in 1866. His improvement of the compass, of the method of measuring electricity and of predicting the level of the tides, were all epoch-making advances. He was elected to membership in nearly all the learned societies of Europe, and elevated to the peerage in 1892. His principal writings are "A treatise on natural philosophy" (with Professor Tait), "On heat and electricity," "Paper on electrostatics and magnetism," "Mathematical and physical papers" (three volumes), "Popular lectures and addresses" (three volumes), and the "Baltimore lectures." He was buried in Westminster Abbey.

DR. ASAPH HALL died November 22, 1907. Born at Goshen, Connecticut, October 15, 1829, he was appointed assistant at the Harvard observatory in 1857, Naval observer in 1862 and professor of mathematics in the Navy from 1863 to 1891, when he retired under the age limit. He was then appointed professor of mathematical astronomy at Harvard University, which position he held until 1901. Dr. Hall was a prolific writer, and his investigations won for him numerous medals and membership in most of the learned societies of America and of Europe. Among his most important discoveries may be mentioned the period of Saturn, two new satellites of Mars, the motion of the line of apsides of Hyperion, and important contributions to the theory and data of stellar parallax.

PROFESSOR C. A. YOUNG, emeritus professor of astronomy at Princeton University, died January 4, at the age of 73 years.

DR. A. LÉVY, professor of mathematics at the municipal industral school of Paris, died December 28, at the age of 64 years.

PROFESSOR T. BARKER, formerly professor of mathematics at Owens College, Manchester, England, died November 20. He bequeathed about £40,000 to Victoria University, to establish professorships and scholarships in mathematics and in botany.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BACHMANN (P.). Grundlehren der neueren Zahlentheorie. (Sammlung Schubert, LIII). Leipzig, Göschen, 1907. 8vo. 11 + 271 pp. Cloth. M. 6.50

BAILEY (F. H.). See WOODS (F. S.).

BENNECKE (F.). Eine konforme Abbildung als zweidimensionale Logarithmentafel zur Rechnung mit komplexen Zahlen. Festschrift des königlichen Victoria-Gymnasiums zur 300-jährigen Jubelfeier des königlichen Joachimthalschen Gymnasiums zu Berlin. Berlin, Salle, 1907. 7 pp. M. 2.00

BRYAN (G. H.) and PINKERTON (R. H.). The elements of the geometry of the conic. London, Dent, 1907. 8vo. 282 pp. 3s. 6d.

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